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EQUIVARIANT SKK AND VECTOR FIELD BORDISM

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We use a notion of equivariant Euler characteristic in order to extend classical results on controllable cutting and pasting, and vector field bordism, to the case of manifolds acted on by an arbitrary finite group G, and modelled on a fixed virtual representation (in the sense of W. Pulikowski and C. Kosniowski). By restricting attention to such G-manifolds, one finds that classical results continue to hold in the oriented and unoriented case. This extends work of several authors.

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Equivariant bordism	equivariant cutting and pasting
vector field bordism	
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1. Introduction and statement of results

Let G be a finite group. The purpose of this note is to explore relationships between nonvanishing G-invariant vector fields on compact smooth G-manifolds (and bordisms between G-manifolds), various notions of equivariant Euler characteristic, and equivariant controllable cutting and pasting of G-manifolds. This is prompted both by the classical relationships between these notions (in the absence of a group action; see for example [6]), and by equivariant results recently obtained by Heithecker [5], Prevot [11] and Komiya [8, 9].

The basic question being addressed is the following. Given two (possibly oriented, in a suitable sense) G-manifolds M and M' which are G-cobordant, determine necessary and sufficient conditions on M and M' in order that:

(a) there exists a bordism Y of M and M' supporting a nowhere zero invariant tangent field inward normal on M and outward normal on M';

(b) M and M' are equivariantly SKK (controllable cut and paste) equivalent. Nonequivariantly, (a) and (b) are equivalent [6], (a) having been first studied by Reinhart [13], and the relevant invariants are Euler characteristic and, in the oriented case, Kervaire semicharacteristic, of M and M'.

In the presence of a group action, the situation is less clear; analogous and partial results apparently hold for abelian groups of odd order, and, in the unoriented case, for involutions. The results of Komiya seem to indicate that the appropriate notion of equivariant Euler characteristic may be a complicated object, breaking up over components of fixed subsets corresponding to different local representations [9].

It is our long-term goal to discover the correct notion (if any) of equivariant Euler characteristic which gives the pertinent information on equivariant SKK equivalence. Here we begin this program by studying smooth G-manifolds modelled locally on a single (virtual) representation, as first considered by Kosniowski [7] and Pulikowski [12]. These restricted forms of G-manifolds possess important technical advantages: first, one has a natural notion of orientability [14, 15]—especially useful in the case of actions by groups of even order; in addition, the fixed-point data of such manifolds seem less pathological than the corresponding data for unrestricted smooth G-manifolds. (See [16] for a discussion.) Examples of such G-manifolds include arbitrary smooth G-manifolds with connected fixed-sets and possessing a stationary subset, as well as arbitary free G-manifolds [14].

We show that, for G-manifolds, oriented or not, modelled on a fixed virtual representation, the classical relationship between vector field bordism and SKK continue to hold equivariantly in the appropriate sense, for arbitrary G. Further, for $|G|$ odd, one has determining invariants corresponding to the classical ones, where the appropriate notion of Euler characteristic emerges as an element of the Burnside ring of G. Our results are as follows. (Precise statements appear in Section 5).

Theorem 1. *M and M' are equivariantly SKK-equivalent iff they are equivariantly vector field cobordant (in the sense of Reinhart).*

Theorem 2. *G-manifolds M and M' (modelled on a fixed virtual representation) which are equivariantly bordant via a bordism with no isolated fixed points are equivariantly SKK-equivalent iff their equivariant Euler characteristics agree (in the Burnside ring). In the oriented case, for $|G|$ odd, they are SKK-equivalent iff their equivariant Euler characteristics and fixed-set Kervaire invariants agree.*

Theorem 1 generalizes [6, Theorem 4.4], while Theorem 2 generalizes [13].

For arbitrary G-manifolds (not necessarily modelled on a fixed representation), the situation is, as indicated above, far more complicated, and the first author has begun a program to attempt to understand the manner in which the local representations interlock [17], this being a first step to the formulation of a theory of global Euler-characteristic type invariants. In such a general setting, it is felt that the results here will form the basis of the more general theorems anticipated, as any smooth G-manifold is a union of G-manifolds modelled on some fixed representation (possibly of a subgroup).

In Section 2, we derive the basic properties of equivariant Euler characteristics, continuing our discussion of invariant vector fields from [18]. In Sections 3 and 4 we prove preliminary technical results leading to proof of the theorems in Section 5.

The authors are indebted to R.E. Stong, who first suggested this approach in private communication, and to A. Assadi, for suggesting the approach we use in Proposition 3.2.

2. Stable and unstable equivariant Euler characteristics

Classically, the Euler characteristic $\chi(M)$ of a smooth compact manifold M emerges as the single obstruction to the existence of a nowhere zero smooth tangent vector field μ on M with μ outward normal on ∂M. It thus seems appropriate to define the equivariant Euler characteristic of a (smooth compact) G-manifold as the analogous obstruction to the existence of a smooth tangent G-invariant vector field on M. In order to make sense of this, we use the notion of canonical transverse regularity of tangent G-vector fields from [18].

Definition 2.1. A smooth tangent G-vector field μ on M is said to be canonically transverse if the following conditions hold on μ.

(i) μ is nonsingular on ∂M;

(ii) the zeros of μ form a discrete G-set $\mathcal{K}(\mu)$ in $\mathring{M}$;

(iii) if $x \in \mathcal{K}(\mu)$ and H is its isotropy subgroup, then, noting that $\mu|M^H$ is locally the sum of two fields, μ^H along M^H and μ_H normal to M^H, one insists that μ_H be outward normal at x and μ^H of index ± 1 there.

By [18, Theorem 1A], such fields always exist on a G-manifold M (henceforth assumed smooth and compact). If μ is a canonically transverse tangent G-vector field on M, the G-set $\mathcal{K}(\mu)$ may be regarded as a virtual G-set (s-t) in M via the signs of the local indices.

Definition 2.2. Let s and s' be two finite G-sets imbedded in M. The sets s and s' will be said to be M-equivalent if there is a one-parameter family (s_t) of G-sets, with $s_0 \cong s$, $s_1 \cong s'$, and each s_t G-equivalent to $s \cong s'$. We then take $A_M(G)$ to be the quotient of the free abelian group on M-equivalence classes of G-sets, modulo relations of the form $[s \cup t] \approx [s+t]$ for G-sets s and t embedded disjointly in M. (The classical Grothendieck construction is inappropriate here, since unions of finite G-sets in M are not necessarily embeddable in M.) This gives $A_M(G)$ the structure of an additive abelian group. Thus two isomorphic G-sets s and s' in M with opposite sign are thus allowed to cancel if s can be continuously moved onto s' in M.

If μ is as above, assume that μ is outward normal on ∂M. $\mathscr{H}(\mu)$ then defines an element $\chi(M) \in A_M(G)$. One has a natural additive homomorphism $s: A_M(G) \to A(G)$, where $A(G)$ is the Burnside ring of G. Denote $s(\chi(M))$ by $e(M)$.

Definition 2.3. The stable and unstable Euler characteristics of a compact smooth G-manifold M are the elements $e(M) \in A(G)$ and $\chi(M) \in A_M(G)$ respectively.

The following lemma ensures that these are both well-defined.

Lemma 2.4. *If μ and μ' are canonically transverse tangent G-vector fields on M and outward normal on ∂M, then $\chi(\mu)$ and $\chi(\mu')$ define the same element in $A_M(G)$.*

Proof. The proof of [18; Proposition 5.1] applies verbatim after the following minor changes. First replace the ambient manifold there, which was the unit disc $D(V)$ of the G-module V, by the manifold M. One may then replace the little discs $D'(V)$ in the proof by copies of $G \times_H D(V)$ for various subgroups H and H-modules V. The argument there now implies the result. □

Remarks 2.5. (i) One may use a smooth G-triangulation [1] of M to obtain a canonically transverse tangent G-vector field μ on M as follows. The field μ is first defined on the interiors of the simplices as being radially inward, with a singularity at each barycenter. This easily extends to a smooth tangent field on M with singular set the set of barycenters. To fix μ on the boundary, add an equivariant collar and an outward component which increases from zero as the boundary is approached. This creates no new singularities, and it is easy to see that the field remains canonically G-transverse after this construction.

(ii) Let $\phi(G)$ denote the set of conjugacy classes of subgroups of G, and let

$$d:\ A(G) \to \prod_{(H) \in \phi(G)} \mathbb{Z}$$

be the ring monomorphism which assigns to a virtual G-set $(s\text{-}t)$ the integer $|s^H| - |t^H|$ in the (H)th coordinate. (See, for example, [3] for a thorough treatment.) Then one has $d(e(M))_{(H)} = \chi(M^H)$, the nonequivariant Euler characteristic of the fixed-set, for each subgroup $H \subset G$, this following immediately from the definition of $\chi(M)$ above. Thus the stable characteristic is entirely specified by the fixed-set data.

(iii) It is now a tautology that M admits a nowhere zero vector field outward normal on ∂M iff $\chi(M) = 0$.

(iv) The stable and unstable characteristics are readily seen to agree when all nonempty fixed-set orbits are connected and when all gaps have magnitude at least 2. Note that the gap condition is automatic in the case of odd-ordered group actions.

The following permits one, in principle, to recover the unstable characteristic through knowledge of more fixed-set data.

Proposition 2.6. *For each $H \subset G$, let M_H denote the union of proper fixed subsets in M^H, and let $M^{(H)} = M^H - M_H$. Let $D(H)$ denote the set of components of $M^{(H)}/NH$, and define*

$$\underline{d}\colon A_M(G) \to \prod_{(H)\in\phi(G)} \prod_{C\in D(H)} \mathbb{Z}$$

by assigning to a virtual G-set (s-t) in $A_M(G)$ the tuple whose $((H), C)$-coordinate is $|C \cap (s \cap M^{(H)})/NH| - |C \cap (t \cap M^{(H)}/NH|$. Then $\underline{d}$ is an injective additive homomorphism. Further, $\underline{d}(\chi(M))$ has (H, C)-coordinate $\chi(C) - \chi(\partial' C)$, where $\partial' C$ is that portion of the boundary of an invariant NDR neighborhood U_H of M_H in M^H which intersects G.C.

Proof. The first assertion is straightforward to verify; one observes that the component data above suffices to specify a virtual G-set up to equivalence in $A_M(G)$. For the second, one notes that, given a canonical tangent G-vector field μ on M, it is possible to construct G-vector fields on the pieces C with the singularities specified by $\underline{d}(\mathscr{K}(\mu))$, but pointing inward on $\partial' C$. Hence the correction terms $\chi(\partial' C)$. $\square$

While the unstable Euler characteristic measures the singularities in a tangent field on M, it will turn out that the stable characteristic suffices if one allows the attaching of handles, as will be appropriate in the study of vector field bordism.

3. Oriented G-manifolds, Bordism and SKK

Here, we assemble basic facts on these topics and prove preliminary results.

As pointed out in the introduction, we shall consider G-manifolds modelled locally on a fixed virtual ambient representation $V - W$, as first discussed by Pulikowski in [11] and by Kosniowski in [7]. Fix finite dimensional orthogonal G-modules V and W, and let M be a smooth G-manifold. Then M is said to have equivariant dimension $(V - W)$ if, for each $x \in \operatorname{int} M$, there is a smooth G_x-equivariant embedding

$$i\colon Y \to M$$

of a G_x-module Y, taking O to x, where $Y + W = V$ as a G_x-module. Note that the boundary of a $(V - W)$-manifold automatically has dimension $(V - W - 1)$, where the integer 1 refers to $\mathbb{R}^1$ with trivial G-action.

The tangent bundle τ_M of M has the property that $\tau_M \oplus W$ has fibers modelled on the representation V. An *orientation of M* is then an orientation of the G-bundle $\tau_M \oplus W$ in the sense of [14, § 1]. The fiber automorphisms corresponding to changes of local coordinates thus have degree $+1$ on each fixed set. Such G-manifolds automatically possess oriented fixed sets, and all the normal data are compatibly oriented.

When W has trivial G-action and M is an oriented $(V-W)$-manifold, any two points in components of M^H with dimension ≥ 1 have neighborhoods that are diffeomorphic through an orientation preserving H-map, for $H \subset G$. As this will be crucial in the arguments to follow, we henceforth assume W trivial in the oriented case, and unrestricted otherwise.

The category of oriented G-manifolds gives rise to associated G-bordism groups Ω_*^G, indexed on RO(G), the real representation ring of G, and defining an equivariant homology theory. (See [14].) If $\mu = [V-W] \in \mathrm{RO}(G)$, then Ω_μ^G is the group of G-bordism classes of oriented G-manifolds with dimension $(V-W)$, where the trace of a G-bordism is required to be a G-manifold of dimension $(V-W+1)$. Further, Ω_*^G has a natural structure as a graded module over the Burnside ring $A(G)$.

An analogous theory exists in the unoriented case, and, although the arguments to follow assume orientability, they apply equally well to unoriented G-manifolds.

We shall require some facts on G-handlebody decompositions of suitable G-bordisms. If Y is an orthogonal (finite dimensional) G-module, denote by $D(Y)$ and $S(Y)$ the unit disc and sphere, respectively, in Y. By abuse, if $k \geq 0$ is an integer, we denote $D(\mathbb{R}^k)$ and $S(\mathbb{R}^k)$ by $D(k)$ and $S(k)$ respectively. If the virtual representation $(V-W)$ is represented H-equivariantly by an H-module Y, we shall sometimes use $D(V-W)$ and $S(V-W)$ to refer to $D(Y)$ and $S(Y)$ as H-spaces.

Definition 3.1. Let M be a $(V-W)$-manifold and let $k \geq 0$. Given a G-embedding

$$\phi\colon\ G \times_H (D(V-W-k) \times S(k+1)) \to M,$$

one may do surgery on ϕ in the usual manner. We refer to such a surgery as a surgery of type $(H, V-W, k)$. In the case of oriented G-manifolds, one insists that ϕ be orientation preserving.

It turns out that a suitable G-cobordism Y between two G-manifolds M and N may be constructed by a sequence of such (special) surgeries, starting with M and attaching successive G-handles of the above type. (Komiya seems, in [8], to wrongly assume this true of arbitrary G-bordisms. In particular, his claim, which does not, as he asserts, follow from the equivariant Morse theory of Field, [4], seems to fail even in the simple case $M = S(V)$ with $V^G = 0$, and $Y = D(V)$; no such sequence of G-handlebody attachments starting at $S(V)$ can realize the single fixed point O.)

Proposition 3.2. *Assume that Y^{V-W+1} is a G-cobordism between the $(V-W)$-manifolds M and M' such that, for each $H \subset G$, and each component C of Y^H, one has $C \cap M \neq \emptyset$. Then there is a sequence of G-manifolds*

$$M = M_0, M_1, \ldots, M_n = M'$$

of dimension $(V-W)$ with each M_i cobordant to M_{i+1} via the trace Y_i of a surgery of type $(H, V-W, k)$, for some k, with $\bigcup_i Y_i = Y$.

Proof. The proposition essentially follows by the methods in [*LR*; Theorem 13], applied to the cobordism Y. We therefore sketch the slight elaboration required. Beginning with an equivariant collar on M, one does induction up orbit types, first decomposing the orbit space of fixed sets by maximal subgroups into handlebodies, beginning at the collar. One then pulls back to a decomposition of the corresponding fixed sets themselves, and thickens the construction via normal tubes. This is seen to yield handlebody attachments of the above type. One then extends over minimal orbits not yet considered, until Y is exhausted. The fact that all fixed set components intersect M allows one to begin the process at M at each stage. $\square$

Definition 3.3. We refer to a G-cobordism of the above type as a nice G-bordism.

Let $\mathscr{F}(V-W)$ be the family of subgroups $H\subset G$ such that $(V-W)$, regarded as a virtual H-module, is represented by an actual H-module U. Then if Y is a G-bordism between M and M', there are no isolated fixed points $x'\in Y$ with isotropy subgroup $H\in\mathscr{F}(V-W)$, since Y is $(V-W+1)$-dimensional. Further, all points in M and M' have isotropy subgroup lying in $\mathscr{F}(V-W)$. Denote by $E\mathscr{F}(V-W)$ the universal G-space associated with the family $\mathscr{F}(V-W)$. The projection

$$\pi\colon\ E\mathscr{F}(V-W)\to \text{point}$$

induces a corresponding map

$$\pi_*\colon\ \Omega_*^G(E\mathscr{F}(V-W))\to\Omega_*^G.$$

Classifying G-maps from M and M' into $E\mathscr{F}(V-W)$ then define associated classes $\{M\}$ and $\{M'\}$ in $\Omega_{V-W}^G(E\mathscr{F}(V-W))$.

Lemma 3.4. *Assume that* $\{M\}=\{M'\}$, *and that* $e(M)=e(M')$. *Then there is a disjoint union* K *of* G-*manifolds of the form* $G\times_H S(V-W+1)$ *with (varying)* $H\in\mathscr{F}(V-W)$ *such that* $M+K$ *and* $M'+K$ *are nicely* G-*cobordant.*

Proof. By the hypothesis, there exists a G-bordism Y of M and M' with no isolated points fixed by any subgroup $H\subset G$, so that all fixed subsets in Y have dimension at least 1. If each component of each fixed subset of Y intersects M, then we are done, choosing $K=\emptyset$. Thus assume H maximal with the property that $Y^H\neq\emptyset$ and $Y^H\cap M=\emptyset$. Add a copy of $G\times_H S(V-W+1)$ to K, and extend Y to a bordism $M+K\approx M'+K$ via a copy of $\{G\times_H S(V-W+1)\}\times I$. Then $K^H\times I$ contains a fixed submanifold of dimension at least 1, and one may attach a tube $\{G\times_H S(V-W+1)\}\times I$ connecting the orbit of an H-fixed point in $K\times I$ to a corresponding orbit in Y^H. This construction, repeated inductively up orbit types, results in a new bordism Y' with all fixed-set components connected to M with the possible exception of 1-dimensional components. This case causes potential problems only when one has a 1-dimensional component of Y^H which intersects M' and not M. However, since we are insisting that $e(M)=e(M')$, there must be another component of Y^H which intersects M and not M'. One can therefore connect these up by attaching a tube of the above type, and we are done. $\square$

One may now consider equivariant controllable cutting and pasting. Fix a virtual G-module $(V-W)$. As in [6], one may factor the semigroup $\mathcal{M}$ of G-diffeomorphism classes of oriented $(V-W)$-manifolds (or of unoriented $(V-W)$-manifolds, depending on context) by all relations of the form

$$N\cup_{\phi}-N'+L\cup_{\mu}-L'\approx L\cup_{\phi}-L'+N\cup_{\mu}-N',$$

where $\partial N=\partial L$, $\partial N'=\partial L'$ and ϕ and μ are (orientation preserving) G-diffeomorphisms $\partial N=\partial L\to\partial N'=\partial L'$. The Grothendieck group of the result will be denoted by $\mathrm{SKK}_{\gamma}^{G}$ in the oriented case and by KK_{γ}^{G} in the unoriented case, where $\gamma=(V-W)$. The equivalence class of a G-manifold M in either theory will be denoted by $[M]$. The SKK groups just obtained possess a natural $A(G)$-module structure; if s is a G-set, one defines $s[M]=[s\times M]$, and observes that this is a well-defined operation which extends to an action by $A(G)$ in the evident way.

As observed by Heithecker in [5] (see also [8]), the work in [6] generalizes directly to give the following.

Lemma 3.5. *Let M' be obtained from M via a G-surgery of type $(H, V-W, k)$. Then*

$$[M]=[M']+(-1)^{k+1}[G\times_{H}S(V-W+1)].$$

Let Y denote the trace of the surgery above. Then

$$e(Y)=e(M)+\{G/H\}-e(G\times_{H}S(k+1)),$$

where $\{G/H\}\in A(G)$ is the element represented by the G-set G/H. Thus

$$e(Y)=e(M)+(-1)^{k}\{G/H\}.$$

In particular, $e(Y)-e(M)\in A(G)$ is represented by a virtual G-set with no (virtual) G-orbits of type G/K unless $K\in\mathcal{F}(V-W)$. Denote by $A_{\mathcal{F}}(G)$ the ideal in $A(G)$ consisting of virtual G-sets expressible as sums of orbits of the form G/K with $K\in\mathcal{F}(V-W)$. If $x\in A_{\mathcal{F}}(G)$, then one defines the element

$$x[S(V-W)]\in\mathrm{SKK}_{\mu}^{G}$$

as follows. Represent x as a sum, $\sum_j n_jG/K_j$ with each $K_j\in\mathcal{F}(V-W)$. Then set

$$x[S(V-W)]=\sum_{j} n_j[G\times_{K_j}S(V-W)].$$

Notice that, if $S(V-W)$ is replaced by any G-manifold, then the definition continues to make sense, and agrees with the action of $A(G)$. In view of this notational gimmick, Lemma 3.1, together with the remarks following, implies the following by an easy induction argument.

Proposition 3.6. If M and M' are nicely G-cobordant with trace Y, then $e(Y)-e(M)\in A_{\mathcal{F}}(G)$, and

$$[M]=[M']-(e(Y)-e(M))[S(V-W+1)].$$

Corollary 3.7. *Let M be any $(V-W+1)$-dimensional closed G-manifold with orbit types G/K with $K \in \mathcal{F}(V-W)$, and assume that there exists an embedding of $D = G\times_H D(V-W+1)$ in M which meets all the fixed subsets in $M-D$. Then $e(M)[S(V-W+1)]=0$.*

Proof. By the hypothesis, $M-D$ is a nice null-bordism of ∂D, whence

$$[\partial D] = 0-(e(M-D)-e(\partial D))[S(V-W+1)].$$

Since $e(M-D)=e(M)-e(D)+e(\partial D)$, the right hand side is $(e(M)-e(D))\times [S(V-W+1)]$. But $e(D)[S(V-W+1)]=[G\times_H S(V-W+1)]=[\partial D]$, by definition of the action of $\{G/H\}$, and the result follows. □

Corollary 3.7 has the following converse.

Lemma 3.8. *Let $x \in A_{\mathcal{F}}(G)$, and assume that $x[S(V-W+1)]=0$. Then there exists a $(V-W+1)$-manifold M with orbit types associated with $\mathcal{F}(V-W)$, and with $x=-e(M)$. Further, if x has a non-zero summand $\{G/H\}$, then M admits a G-embedding of $G\times_H D(V-W+1)$.*

Proof. One essentially mimics the construction in the proof of [6, Theorem 4.2], as follows. First, let x be represented by the virtual G-set x^+-x^-. Since $x[S(V-W+1)]=0$, one has

$$\begin{aligned} &x^- S(V-W+1)+N\cup_\phi - N' + L\cup_\mu - L' \\ &\quad = x^+ S(V-W+1)+L\cup_\phi - L' + N\cup_\mu - N', \end{aligned}$$

for suitable N, N', L, L', where the G-manifold $x^\pm S(V-W+1)$ is given the evident interpretation. To construct M, one takes $Y(N)$ to be the union of $N\times[0,1]$ and $N'\times[0,1]$, glued as in the diagram in [6, p. 47], and similarly for $Y(L)$. Denote by $\pi(x^+)$ the boundary of $\gamma(x^+)=x^+S(V-W+1)$, and similarly for $\pi(x^-)$. $\gamma(N)-Y(L)+\pi(x^-)-\pi(X^+)$ then has boundary

$$\begin{aligned} &\gamma(x^-)-\gamma(x^+)+N\cup_\phi - N'+L\cup_\mu - L' - (N\cup_\mu - N' + L\cup_\phi - L') + T - T \\ &\quad = \gamma(x^+)-\gamma(x^+)+N\cup_\mu - N + L_\phi - L' \\ &\qquad - (N\cup_\mu - N' + L\cup_\phi - L') + T - T \end{aligned}$$

where T is the mapping torus of $\mu\phi^{-1}$. We can therefore identify the boundary components pairwise to obtain a $(V-W+1)$-dimensional manifold P. A direct calculation shows that

$$e(P) = x^+ + x^- - e(\gamma(x^+)) - 2e(\partial N).$$

Since $\partial N\times S^2$ is a $(V-W+1)$-dimensional manifold with stable Euler characteristic $2e(\partial N)$, one has, with $R=p+\partial N\times S^2$,

$$e(P) = x^+ + x^- - e(\gamma(x^+)).$$

For the finite G-set s, let $\rho(s)$ be the torus bundle $\gamma(s)\times S^1$ over s, so that $\rho(s)$ has dimension $(V-W+1)$. The G-set x^+ may be associated with an embedding in R, and it certainly also embeds in $\rho(x^+)$. Now replace R by the connected sum,

$$M = R\#_{(x^+)}\rho(s^+),$$

over deleted disc neighborhoods of the embeddings of x^+. We now claim that $e(M) = -x$. Indeed, the Euler characteristic of M^H is given by

$$d_H\{x^+ + x^- - 2x^+\} = d_H(-x);$$

if $\dim M^H$ is even, the -2 term comes from $\gamma(x^+)$, and when $\dim V^H$ is odd, it comes from $\rho(x^+)$, thus $e(M) = -x$, as required. □

Remarks 3.9. The fact that we are considering $(V-W)$-manifolds, as opposed to G-manifolds in general, is crucial here. Allowing the representation to vary forces one to consider each local representation separately. This, as mentioned in the introduction, has prompted an ongoing project in the study of interlocking representations, [17].

4. Reinhart bordism

In [13], Reinhart introduced the notion of bordism with vector fields; two manifolds M and M' are Reinhart cobordant if there is a cobordism Y of M and M' and a nowhere zero tangent vector field v on Y with v inward normal on M and outward normal on M'. (We shall refer to such a field as a Reinhart vector field.) This is strongly related to SKK in view of [6, Theorem 4.4]; two manifolds M and M' are Reinhart cobordant iff they are equivalent in SKK_*, with an analogous statement holding in the unoriented case. This in turn is related to the Euler characteristic; M and M' are Reinhart cobordant iff they are cobordant, $\chi(M) = \chi(M')$, and, in the oriented case, if the Kervaire semicharacteristics of M and M' agree.

Equivariantly, one insists that all vector fields be invariant, and one has partial results (for example, those of Heithecker [5] in the oriented case for G an abelian group of odd order, and Komiya [7] in the unoriented case under various hypotheses).

Here we establish preliminary results required for the proof of Theorems 1 and 2. Henceforth, M and M' will be assumed to be (oriented) $(V-W)$-manifolds, and all G-bordisms will be assumed to have dimension $(V-W+1)$.

Lemma 4.1. *Let v be a canonically transverse tangent G-vector field on M, and assume that, for each $H \subset G$, the zeros of $v|M^H$ formally cancel in pairs. Then, by combining the operations of*

(i) *attaching G-handles corresponding to embeddings of the form*

$$\phi\colon\ G \times_H (D(V-W) \times S(1)) \to M,$$

where the discs fall on small disc neighborhoods of the singularities,

(ii) *attaching G-handles of the form* $\{G\times_H (S^1\times D(V-W-1))\}\times I$ *by using embeddings of* $\{G\times_H (S^1\times D(V-W-1))\}\times S(1)$ *within small disc neighborhoods of the singularities, and*

(iii) *adding disjoint spheres of the form* $G\times_H S(V-W+1)$, *one can remove the singularities.* (*Note that we are not assuming the fixed sets of M to be connected.*)

Proof. Since the field is canonically transverse, the indices at its (isolated) zeros are entirely carried by restriction to fixed sets by appropriate isotropy subgroups. Thus, if $H\subset G$, and if $x\in M^H$ is a zero with isotropy subgroup H, then its equivariant index, as an element of $A(H)$, is ± 1 according as the index of $v|M^H$ is ± 1. Our argument is by induction up orbit types; at each stage we attach handles which have the effect of removing the singularities of that orbit type, and (possibly) replacing them by new and neighboring singularities of strictly larger orbit type. At the conclusion of each inductive step, one is again left with a canonically transverse field satisfying the hypothesis of the lemma, thus enabling induction to continue.

Thus let H be the subgroup corresponding to the present stage of induction, and let x and y be a cancelling pair of singularities in $M^{(H)}=M^H-M_H$, with the index of x taking the value $1\in A(H)$, and directed radially outward away from x. We consider several cases. Denote by $a(x)$ the local singularity at x, but with the field direction reversed in the radial direction.

Case (i). x and $a(x)$ have the opposite index on M^H.

Here, one must have dim M^H odd, and $a(x)$ has index $-1+\sum_j n_j[H/K_j]$ in $A(H)$, with each K_j a proper subgroup of H. Further, the fact that one can make the field $a(x)$ on $D(V-W)$ canonically G-transverse implies that each of the H/K_j may be embedded as orbits in $D(V-W)$. Thus, since

$$\mathrm{ind}(y)+\sum_j n_j[H/K_j]=\mathrm{ind}(a(x)),$$

one may, using [18, Theorem 1A], deform the field near y H-equivariantly to one with singularity a copy of $a(x)$ at y, and canonical singularities on an H-set corresponding to the embedded orbits. Since now the fields near the singularities at x and y are radially opposite, one may remove them simultaneously by attaching a narrow tube $S(V-W)\times I$, removing the copies of $D(V-W)$ it bounds, and extending the field tangentially and nonsingularly along the tube. The G-action now permits one to repeat the process at the points of $G.x$ and $G.y$, obtaining the G-manifold M'. If U is an invariant neighborhood of the region effected by this procedure, an easy calculation shows that we have decreased $e(U)$ by $e(G\times_H S(V-W+1))$. Thus taking the disjoint union of M' with $R=G\times_H S(V-W+1)$ leaves $e(U)$ unchanged. Since here, dim $S(V-W+1)^H$ is odd, there exists a (canonically transverse) H-vector field on R which is nonsingular on R^H. Thus $M'\cup R$ has no new singularities in the H-fixed set (or any proper fixed subsets). Futher, since the local stable Euler characteristic is unchanged, a consideration of fixed sets shows that the singularities within U, (as modified by addition of R), continue to cancel formally as in the hypothesis. This completes the inductive step in this case.

Case (ii). x and $a(x)$ have the same index on M^H.

Here, dim M^H is even. Further, one cannot have dim $M^H = 0$, since then x and y must both have index $+1$ in $A(H)$, by canonical transverse regularity, contradicting our assumptions. Choose small disc neighborhoods Dx and Dy of x and y respectively, and embed in each a copy of $S^1 \times D(V - W - 1)$, with the embeddings of S^1 centered at x and y, this being possible since dim $M^H \geqslant 2$. The result of attaching a handle according to (ii) above leaves $e(M)$ unchanged locally (in the sense described in Case (i)). Further, since we have not changed the local Euler characteristic, one may extend the field over the H-fixed set of the handle with no singularities. Further, the process leaves x and y in the same component of the H-fixed set. One may therefore alter the field on a small disc neighborhood of the singularities x and y in M^H to become zero-free there, without effecting the field on the boundary of that neighborhood. Now extend the existing field H-equivariantly in an arbitrary fashion over the handle, and make it consistently transverse regular. The resulting field now has the desired properties for induction, and we are done. □

Lemma 4.2. (i) *Let Y be a G-bordism between M and M' admitting a Reinhart field. Then $e(M) = e(M') = e(Y)$.*

(ii) *Assume that there exists a G-bordism Y between M and M' with $e(M) = e(M') = e(Y)$. Then M and M' are Reinhart G-bordant (note necessarily via Y).*

Proof. For (i), assume that Y admits a Reinhart field v. Then, for each $H \subset G$, v restricts to a Reinhart field on Y^H, whence $\chi(M^H) = \chi(M'^H) = \chi(Y^H)$, by [13]. The result now follows by the definition of the stable equivariant Euler characteristic e.

Turning to (ii), assume that one has a G-bordism Y with $e(M) = e(M') = e(Y)$. Then, for each $H \subset G$, one has $\chi(M^H) = \chi(M'^H) = \chi(Y^H)$. One may now construct a tangent G-vector field v inward normal on M and outward normal on N, and then equivariantly deform v to a field (which we again denote by v), and which is canonically transverse regular, as in Section 2. It follows by [13] that, for each $H \subset G$, the index of $v | Y^H$ is zero, so that the local indices cancel in pairs. Now apply Lemma 4.1 to obtain the desired nonvanishing field. □

5. Proof of Theorems 1 and 2

We continue to assume that M and M' are $(V - W)$-dimensional (oriented) G-manifolds, and that all G-cobordisms are through $(V - W + 1)$-dimensional manifolds. Recall that $[M]$ refers to the SKK-class of the G-manifold M.

Theorem 1. *M and M' are Reinhart bordant iff $[M] = [M']$.*

Proof. If M and M' are Reinhart bordant, then, by Lemma 4.2, one has, for a G-bordism Y admitting a Reinhart field, $e(M) = e(M') = e(Y)$. Further, since Y

admits a nowhere zero field, Y has no G-orbits G/K unless $K \in \mathscr{F}(V-W)$ (or else there would be isolated zeros), whence $\{M\}=\{M'\}$ in the notation of Section 3. By Lemma 3.4, $M+K$ and $M'+K$ are nicely G-cobordant for suitable K, with $e(M+K)=e(M'+K)$. Let Y' denote the cobordism so formed. One now has, by direct calculation,

$$\begin{aligned}
e(Y') &= e(Y)+\sum [e(G\times_H S(V-W+1))-e(G\times_H S(V-W+2))] \\
&\quad -\sum e(G\times_H S(V-W+2)) \\
&= e(Y)+e(K)-\sum e(G\times_H S(V-W+2))-\sum e(G\times_H S(V-W+2)) \\
&= e(Y)+e(K)-\sum e_1-\sum e_2, \quad \text{say,}
\end{aligned}$$

where the first sum is taken over the summands of K and where the second sum corresponds to the special case of 1-dimensional fixed sets. Note that the terms e_1 and e_2 are Euler characteristics of G-manifolds of dimension $(V-W+1)$ which satisfy the hypothesis of Corollary 3.7. It follows that

$$\begin{aligned}
&(e(Y')-e(M+K))[S(V-W+1)] \\
&\quad = (e(Y)-e(M)-\sum e_1-\sum e_2)[S(V-W+1)]=0,
\end{aligned}$$

since $e(Y)=e(M)$, and by Corollary 3.7. It now follows by Proposition 3.6 that $[M+K]=[M'+K]$, whence $[M]=[M']$, as required.

Conversely, assume that $[M]=[M']$. Since equivariant cutting and pasting restricts to cutting and pasting on fixed sets, one has, by [6, Theorem 4.4], $x(M^H)=x(M'^H)$ for each $H\subset G$, so that $e(M)=e(M')$. Further, the equivariant analogue of [6, Theorem 4.1] shows that M and M' are G-bordant via a $(V-W+1)$-manifold Y containing only orbits derived from $\mathscr{F}(V-W)$. Thus, by Lemma 3.4, one has $N=M+K$ and $N'=M'+K$ nicely G-cobordant via some $(V-W+1)$-manifold Z. It follows that

$$(e(Z)-e(N))[S(V-W+1)]=0$$

by Proposition 3.6. By Lemma 3.8, one now has $e(Z)-e(N)=-e(L^{V-W+1})$ for a suitable L. If $e(L)=0$, we are done by Lemma 4.2(ii). Thus assume that $e(L)$ has a non-zero summand $\{G/H\}$ for some maximal $H\in\mathscr{F}(V-W)$. By the construction of L, there is a G-embedding of $G\times_H D(V-W+1)$ in L. Further, we claim that a similar embedding exists in Z. Indeed, since $e(Z)-e(N)$ is represented by the set of zeros of a tangent field on Z outward normal on N' and inward normal on N, and since $\{G/H\}$ is a summand, there is such an orbit in Z. One can therefore glue L to Z via a G-handle of the form $I\times\{G\times_H S(V-W+1)\}$, obtaining a new G-bordism Z'. Further, one has

$$\begin{aligned}
e(Z') &= e(Z)+e(L)-e(G\times_H S(V-W+2)) \quad \text{by direct calculation,} \\
&= e(N)-e(G\times_H S(W-V+2)).
\end{aligned}$$

The disjoint union Z'' of Z' and a copy of $G\times_H S(W-V+2)$ thus has Euler characteristic equal to that of N. It now follows by Lemma 4.2(ii) that N and N' are Reinhart G-bordant via some $(V-W+1)$-manifold X. Since K is a union of G-manifolds of the form $G\times_H S(V-W+1)$, with the resulting Reinhart field inward normal at one end and outward normal on the other, one may now glue the two end copies of K together and extend the field nonvanishingly over the glueing. This gives the desired Reinhart G-bordism between M and M', as required. □

Theorem 2. (a) *Equivariant stable Euler characteristic and unoriented equivariant bordism in $\mathfrak{N}^G_{V-W}(E\mathscr{F}(V-W))$ are the determining invariants of both unoriented Reinhart G-bordism and unoriented equivariant SKK in dimension $V-W$.*

(b) *If G has odd order, then equivariant stable Euler characteristic, oriented equivariant bordism in $\Omega^G_{V-W}(E\mathscr{F}(V-W))$, and Kervaire semicharacteristic of fixed sets are the determining invariants of both oriented Reinhart G-bordism and oriented SKK equivalence in dimension $V-W$.*

Proof. We show (b), the corresponding argument for (a) (whether or not G has odd order) being an easy adaptation. That the Euler and Kervaire indices are invariants is an immediate consequence of passage to fixed subsets, and that Reinhart cobordisms cannot contain isolated fixed points by any $H\subset G$ is immediate from the invariance of the vector fields we are considering. Thus assume that M and M' agree under these invariants. Start with an arbitrary G-bordism Z between M and M' (with restricted orbit types as hypothesised). Then, by Lemma 3.4, one has, as in the proof of Theorem 1, nicely G-bordant manifolds N and N', with nice bordism Y. By Proposition 3.6, $[N]-[N']=x[S(V-W+1)]$, for some $x\in A_{\mathscr{F}}(G)$. Passing to fixed subsets, one has $[N^H]=[N'^H]$ for each $H\subset G$, since the requisite invariants of N and N' agree on fixed sets. Thus, $\chi(xS(V-W+1)^H)=0$, and, in the case of fixed sets of dimension $1+4k$, $\kappa(xS(V-W+1)^H)=0$. These invariants agree with $d(x^H)\chi(S(V-W+1)^H)$ and $d(x^H)\kappa(S(V-W+1)^H)$ respectively. One must therefore have, in the case of even dimensional fixed sets, $d(x^H)=0$. If the dimension m of the fixed subset is odd and of the form $1+4k$, the condition on κ ensures that $d(x^H)$ is even.

We now wish to alter the G-bordism Y so as to obtain a new, not necessarily nice, $(V-W+1)$-dimensional G-bordism Y' with $e(N)-e(Y')=0$. We do this inductively up orbit-types starting with orbits which are summands of x corresponding to maximal subgroups. Thus let $n\rho$ be such a maximal summand, with $n\rho$ isomorphic to $|n|$ copies of G/H (as represented by a virtual G-set), and let $m=\dim V^H$. If m is even, then, since ρ is maximal, $d(x^H)=n|\rho^H|=0$, so that $n=0$. If $m=4k+1$, then $d(x^H)=n|\rho^H|$ is even. Since G has odd order, this implies that n is even. (Note that the dimension of Y^H must be even here.) Further, $n\rho$ must embed equivariantly in Y, (since x came from a G-set in Y). We can thus take the connected sum of Y with the $(V-W+1)$-dimensional manifold

$$(|n|/2)(G\times_H S(V-W+1)\times S^1)$$

if $n<0$, thereby increasing $d(x^H)=d((e(N)-e(Y))^H)$ by $n|\rho^H|$. If $n>0$, we can increase $d(e(Y)^H)$, and hence decrease $d(x^H)$, by $n|\rho^H|$ by adding a disjoint copy of the $(V-W+1)$-manifold $(n/2)(G\times_H S(V-W+2))$ to Y, thereby completing the inductive step in this case. If $m=4k+3$ and if n is even, the argument in the case $m=4k+1$ works equally well. Note also that this process leaves the Euler characteristics of odd-dimensional fixed subsets of Y unaffected, and preserves the parity of the even-dimensional ones, thus permitting the induction to continue.

Thus we are left with the case $m=4k+3$ and n odd. It suffices to alter Y in such a way as to replace $n\rho$ by $q\rho$ with q even. Let

$$D=\mathbb{C}P^{2k+3}\times D(V-W)(H),$$

where $D(V-W)(H)$ is the orthogonal complement of the H-module $(V-W)^H$ in $(V-W)$. Then $G\times_H D$ has dimension $(V-W)+1$. The obstruction theory in [16] shows that there exists an odd integer N with $N\partial(G\times_H D)$ oriented null-bordant through a $(V-W+1)$-manifold Z with $Z^H=\phi$. If one now glues Z to N copies of $G\times_H D$ along their common boundary, one then obtains a $(V-W+1)$-dimensional G-manifold P with H-fixed set N copies of $\mathbb{C}P^{2k+3}$. Since this has odd Euler characteristic, the disjoint union of Y and P now has $n\rho$ replaced by $q\rho$ with q even. Further, if we take instead Y' to be the connected sum of Y and P over disc neighborhoods of N copies of G/H, the effect on parity is the same (modulo effects on larger orbit summands of $e(Y)$). This, however, leaves the component of the present bordism Y as nice. Indeed, the present Y may be assumed inductively to be the union of a nice bordism Y'' and disjoint spheres, and P is connected to Y'', leaving it nice. Thus the parity of summands of $e(Y)$ corresponding to fixed-set dimensions $4k+1$ are unaffected. Further, the summands corresponding to even fixed-set dimensions remain 0, thus permitting the induction to continue.

This process leaves us with a G-bordism Y' such that $e(N)-e(Y')=0$. Thus, since $e(N)=e(N')$ as well, N and N' are oriented Reinhart G-bordant. That M and M' are now oriented Reinhart G-bordant now follows by the same trick (of closing the copies of K) as was used in the proof of Theorem 1. $\square$

Remarks 5.1. (i) The requirement that G have odd order seems essential to our arguments at each turn. As is common in even order group-actions, the existence of 2-torsion gets in the way of a similar approach for groups of even order. Indeed, it is not hard to see that, if the relevant invariants (κ and e) of M and M' agree, then $2([M]-[M'])=0$. Thus, in the case of actions by groups of even order, these invariants need not detect oriented SKK modulo elements of order 2.

(ii) That the bordism condition is necessary in view of the other conditions is shown as follows. Let $G=\mathbb{Z}/p$ with p an odd prime. Results of Conner and Floyd in [2] show that, if V is any representation of G with $V^G=0$, then the order of $[S(V)]$ in free equivariant bordism is divisible by p. Thus, even though $S(V)+S(V)$ has zero Euler characteristic and Kervaire semicharacteristic, no equivariant V-dimensional null bordism of this manifold is free of isolated fixed points, and hence of vector field singularities, showing that the (stronger) bordism condition cannot be dropped. In the case of non-cyclic groups G of add order, Stong has shown that all nonabelian groups of odd order do admit null bordisms of unit spheres free of isolated fixed points. However, these are not V-dimensional bordisms for any V, and results of the first author, [16], extend the Conner-Floyd result to arbitrary framed V-manifolds, thereby providing counterexamples for all groups of odd order.

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